AIAA JOURNAL
Vol. 38, No. 2, February 2000

Theory of Cylindrical Sandwich Shells with Dissimilar Facings Subjected to Thermomechanical Loads

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A theory is outlined of sandwich box-type composite shells designed to withstand a combination of thermal loading, internal pressure, torsional and axial loads. A cross section of the shell represents a rectangular box with curved cylindrical sections at the corners. The facings of the shell are dissimilar to maximize their efficiency, according to the loads acting on each facing. This approach enables a designer to optimize the structure by maximizing the load-carrying capacity or minimizing the weight. The formulation includes the following developments: 1) global theory of a sandwich shell composed of rectangular and cylindrical sections, where equations of motion are formulated based on a first-order shear deformable version of Sanders's shell theory; 2) theory for local deformations and stresses in the facings, where the facing is treated as a thin geometrically nonlinear plate or shell on an elastic foundation using von Kármán's approach and the elastic foundation represents a support provided by the opposite facing and the core; and 3) an outline of an enhanced micromechanical constitutive formulation based on the incorporation of the effect of the thermomechanical coupling on the material properties and temperature.

Nomenclature

A_{ij}, B_{ij}, D_{ij}	= extensional, coupling, and bending stiffnesses, respectively
C_σ	= specific heat at constant stress
E_m	= modulus of elasticity of matrix
h	= total shell thickness
h_{fi} $(i = 1, 2)$, h_c	= thicknesses of facings and core
k_i $(i = 1, 2, 3)$	= stiffness coefficients of nonlinear elastic foundation
k_{44}, k_{55}	= shear correction factors
M_x, M_y, M_{xy}	= stress couples
N_x, N_y, N_{xy}	= in-surface stress resultants
p	= normal pressure
Q_{ij}	= transformed reduced stiffnesses
Q_x, Q_y	= transverse shear-stress resultants
R	= middle surface radius
s	= sum of principal stresses
T	= temperature (in excess of a reference thermal stress-free value)
T_{gw}, T_{g0}	= glass transition temperatures (wet and dry, respectively)
u, v, w	= in-surface and transverse displacements (in the directions x, y, z)
V_i	= volume fraction (fiber: $i = f$ or matrix: $i = m$)
x, y	= in-surface coordinates (axial and circumferential, respectively)
z	= transverse (radial) coordinate counted from the middle surface of shell or from the middle surface of facing
α_x, α_y, α_{xy}	= coefficients of thermal expansion
α_0	= coefficient of thermal expansion (at reference temperature and at zero stress)
γ_{xy}^o	= shear strain of reference surface
γ_{xz}, γ_{yz}	= transverse shear strains
ε_x^o, ε_y^o	= extensional strains of reference surface
κ_x, κ_y, κ_{xy}	= changes of curvature and twist
ν	= Poisson's ratio
ρ	= mass density
ρ_0, ρ_1, ρ_2	= inertial coefficients in shear-deformable formulation
ψ_x, ψ_y	= rotations about in-surface axes

Subscripts and Superscripts

f	= fiber
m	= matrix

Introduction

DEVELOPMENT of advanced composite structures for aerospace applications has placed sandwich configurations at the leading edge of new technologies. In particular, a sandwich fuselage design represents an innovative concept worth investigating. Examples of applications of sandwich structures include helicopter fuselages, Beech "Starship" and Raytheon "Premier I" airplanes. Although circular cylindrical shells have been the primary issue in sandwich-shell research, alternative shapes may present advantages because of a higher useful volume capacity. In the present paper a box-type sandwich structure is considered (Fig. 1). The loads can include torsion, axial stresses, and pressure as well as a nonuniform thermal field.

The studies of shell sandwich structures have been undertaken by numerous investigators. Mentioned here are the papers of Reissner,[1,2] Bieniek and Fruedenthal,[3] Baker and Herrmann,[4] Kollar,[5] Greenberg et al.,[6] and Frostig.[7] In particular, the latter paper contains a review of work on curved sandwich panels and shells. A comprehensive review of the issues related to mechanics of sandwich-shells has recently been published by Noor et al.[8] Usually, research of sandwich-shells concerns cylindrical or shallow configurations. The studies of box-type shells are typically confined to prismatic structures formed from flat plates.[9,10] The present paper illustrates a first-order theory of shear-deformable cylindrical sandwich shells with dissimilar facings. The analysis is based on the Sanders' improved first approximation theory.[11] Contrary to the work of Greenberg et al.,[6] who considered sandwich shells with dissimilar facings using the Reissner shell theory, the present paper incorporates the effect of in-plane shear stress couples on the in-surface force equilibrium equations, according to the Sanders theory. Other differences are related to including thermal effects

Received 9 September 1998; revision received 12 June 1999; accepted for publication 18 June 1999.

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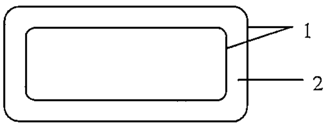

Fig. 1 Composite sandwich box: 1, facings, and 2, core.

and explicitly formulating the governing equations in terms of displacements. The latter formulation is necessary for the analysis of shells consisting of several sections where the continuity conditions along the junctions have to be satisfied. Note that Hsu et al.[12] presented a thermomechanical analysis of shear deformable cylindrical shells based on the shear-deformable version of the Sanders theory. However, their analysis was confined to cross-ply symmetrically laminated shells, whereas the present paper deals with the case of dissimilar facings where the matrices of coupling and bending stiffnesses are fully populated.

Although sandwich shell usually experiences geometrically small global deformations so that it may be characterized by one of shear-deformation theories, local response of the facings should also be analyzed. This is particularly important if the facings are subject to local loads as well as in the case of a box-type cross section considered in this paper where it is necessary to impose appropriate continuity conditions on the individual facings. The facings of sandwich structures found in typical applications are usually sufficiently thin to justify an application of a geometrically nonlinear shell theory. Accordingly, the paper presents a formulation for a geometrically nonlinear analysis of a thin facing, based on the Sanders theory.[13] The support provided by the opposite facing and the core is incorporated by introducing an elastic foundation with the stiffness determined using the methodology discussed in the paper.

The analysis presented here is concerned with shells subjected to a combination of mechanical and thermal loads. The influence of temperature on the material properties has been known for a long time; mentioned here is the work of Chamis,[14] who suggested an analytical formula that could be used to account for the effects of temperature and moisture on the material constants of polymeric matrices. The effect of stress on the coefficient of thermal expansion has also been investigated.[15–17] In addition, temperature within a material was shown to be affected by the local state of stresses; a review of thermomechanical coupling phenomena mentioned here can be found in a recent paper of Dunn.[18] In the present paper the approach that can account for the effects of thermomechanical coupling on the properties and temperature of facings and core materials is outlined. Whereas a detailed analysis will be presented in future publications, it will be conducted using the methodology suggested in this paper.

Linear First-Order Shear-Deformation Theory of Cylindrical Shells

The following formulation is based on the "improved first-approximation" Sanders shell theory.[11] According to this theory, the equations of motion can be presented as

$$
\begin{aligned}
N_{x,x} + N_{xy,y} - \underline{(1/2R)M_{xy,y}} &= \rho_0 \ddot{u} + \rho_1 \ddot{\psi}_x \\
N_{xy,x} + N_{y,y} + \underline{(1/2R)M_{xy,x}} + Q_y/R &= \rho_0 \ddot{v} + \rho_1 \ddot{\psi}_y \\
Q_{x,x} + Q_{y,y} - N_y/R &= \rho_0 \ddot{w} - p \\
M_{x,x} + M_{xy,y} - Q_x &= \rho_1 \ddot{u} + \rho_2 \ddot{\psi}_x \\
M_{xy,x} + M_{y,y} - Q_y &= \rho_1 \ddot{v} + \rho_2 \ddot{\psi}_y
\end{aligned} \tag{1}
$$

The inertial coefficients on the right-hand sides of Eqs. (1) are

$$
\{\rho_0, \rho_1, \rho_2\} = \int_{-h/2}^{h/2} \rho\{1, z, z^2\}\mathrm{d}z \tag{2}
$$

where $h = h_{f1} + h_{f2} + h_c$. Note that the terms underlined in the first two equations disappear if rotations about a normal to the shell surface are neglected.[13] These terms are usually disregarded in nonlinear formulations.[19] The reason for this simplification is related to a much higher in-surface stiffness of the shell as compared to its stiffness in the planes perpendicular to the middle surface.

The kinematic relations corresponding to a first-order shear deformation theory read

$$
\begin{gathered}
\varepsilon_x = \varepsilon_x^o + z\kappa_x, \qquad \varepsilon_x^o = u_{,x}, \qquad \varepsilon_y = \varepsilon_y^o + z\kappa_y \\
\varepsilon_y^o = v_{,y} + w/R, \qquad \gamma_{xy} = \gamma_{xy}^o + 2z\kappa_{xy}, \qquad \gamma_{xy}^o = u_{,y} + v_{,x} \\
\kappa_x = \psi_{x,x}, \qquad \gamma_{xz} = \psi_x + w_{,x}, \qquad \kappa_y = \psi_{y,y} \\
\gamma_{yz} = \psi_y + w_{,y} - v/R \\
2\kappa_{xy} = \psi_{x,y} + \psi_{y,x} + (1/2R)(v_{,x} - u_{,y})
\end{gathered} \tag{3}
$$

The analysis is based on the assumption that the facings operate in the state of plane stress, whereas the core can only resist transverse shearing deformations. Normal stresses acting in the core in the thickness direction are disregarded in the analysis of the sandwich global response. Note, however, that the analysis of individual facings considered in the next section is based on the assumption that normal forces are transferred between the facings, i.e., the core serves as a conductor of these forces, possibly participating in resisting deformations. There is no contradiction in these assumptions because the analysis of a sandwich structure is conducted without accounting for normal stresses in the thickness direction by either first-order or higher-order theories. However, the interaction between individual facings can only be analyzed if the stiffness of the core in the normal (thickness) direction is incorporated into the analysis. The constitutive relations are formulated below for a shell with dissimilar facings by the assumption that each facing is formed from regular symmetrically laminated layers. In practical configurations a facing consists of 0- and 90-deg layers, and layers are oriented at angles $+\theta$ and $-\theta$, with respect to the x axis. The number of $+\theta$ and $-\theta$ layers is always equal. This implies that $A_{16} = A_{26} = 0$, although all other elements of the matrices of stiffnesses are present. In the presence of temperature, the constitutive relations become

$$
\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{Bmatrix}
=
\begin{bmatrix}
A_{11} & A_{12} & 0 & B_{11} & B_{12} & B_{16} \\
A_{12} & A_{22} & 0 & B_{12} & B_{22} & B_{26} \\
0 & 0 & A_{66} & B_{16} & B_{26} & B_{66} \\
 & & & D_{11} & D_{12} & D_{26} \\
 & \text{sym} & & D_{12} & D_{22} & D_{26} \\
 & & & D_{16} & D_{26} & D_{66}
\end{bmatrix}
\begin{Bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \gamma_{xy}^o \\ \kappa_x \\ \kappa_y \\ 2\kappa_{xy} \end{Bmatrix}
-
\begin{Bmatrix} N_x^T \\ N_y^T \\ N_{xy}^T \\ M_x^T \\ M_y^T \\ M_{xy}^T \end{Bmatrix}
$$

$$
\begin{Bmatrix} Q_x \\ Q_y \end{Bmatrix} = \begin{bmatrix} k_{55}A_{55} & 0 \\ 0 & k_{44}A_{44} \end{bmatrix} \begin{Bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} \tag{4}
$$

where the stiffnesses are given by

$$
\begin{gathered}
\{A_{ij}, B_{ij}, D_{ij}\} = \int_{h_{f1}+h_{f2}} Q_{ij}\{1, z, z^2\}\mathrm{d}z, \qquad i, j = 1, 2, 6 \\
A_{rr} = \int_{h_c} Q_{rr}\,\mathrm{d}z, \qquad r = 4, 5
\end{gathered} \tag{5}
$$

The shear correction factors k_{44} and k_{55} for laminated structures can be taken equal to $\frac{5}{6}$ (Ref. 20) or $\pi^2/12$ (Ref. 21). The values of the shear correction factors for sandwich structures were investigated by a number of researchers. Whitney[22] considered sandwich panels with unidirectional graphite/epoxy facings and found the shear correction factors in the planes of orthotropy equal to 0.4098 and 0.6915. However, usually these factors for sandwich structures are close to unity. For example, Yu[23] determined these factors by comparing the frequencies of infinite sandwich plates obtained by the theory of elasticity and the first-order theory and found them in the range of 0.988–1.00. Note that Greenberg and Stavsky[24] observed that the results obtained for cylindrical shells are little affected by small variations in the values of the shear correction factors.

Thermal terms in Eqs. (4) are

$$\{N_x^T, M_x^T\} = \int_{h_{f1}+h_{f2}} (Q_{11}\alpha_x + Q_{12}\alpha_y + Q_{16}\alpha_{xy})\{1, z\}\mathrm{d}z$$

$$\{N_y^T, M_y^T\} = \int_{h_{f1}+h_{f2}} (Q_{12}\alpha_x + Q_{22}\alpha_y + Q_{26}\alpha_{xy})\{1, z\}\mathrm{d}z$$

$$N_{xy}^T = M_{xy}^T = 0 \tag{6}$$

where the integration is carried out throughout the thickness of both facings.

The solution must satisfy the boundary conditions along the curved edges $x = 0$ and L and the continuity conditions along the straight edges of the sections of the shell. The latter conditions will be satisfied in the future finite element analysis based on the present formulation. The former conditions for a simply supported shell subjected to prescribed axial and shear loads are

$$x = 0, \qquad x = L: \qquad w = M_x = \psi_y = 0$$

$$N_x = \bar{N}_x, \qquad N_{xy} + (3/2R)M_{xy} = \bar{N}_{xy} \tag{7}$$

where an overbar indicates an applied load.

Substitution of the kinematic relations (3) into the constitutive relations (4) and the subsequent substitution of these relations into the equations of motion yield the set of five linear differential equations:

$$[L]\{\boldsymbol{U}\} = [I]\{\ddot{\boldsymbol{U}}\} + \{\boldsymbol{P}\} \tag{8}$$

where $[L]$ is the matrix of linear differential operators presented in Appendix A and $\{\boldsymbol{U}\}$ is the vector of displacements (and rotations):

$$\{\boldsymbol{U}\} = \{u \quad v \quad w \quad \psi_x \quad \psi_y\}^T \tag{9}$$

On the right-hand side of Eqs. (8), $[I]$ is the matrix of inertial coefficients and $\{\boldsymbol{P}\}$ is the vector of constant loading terms, including thermally induced contributions given by

$$[I] = \begin{bmatrix} \rho_0 & 0 & 0 & \rho_1 & 0 \\ 0 & \rho_0 & 0 & 0 & \rho_1 \\ 0 & 0 & \rho_0 & 0 & 0 \\ \rho_1 & 0 & 0 & \rho_2 & 0 \\ 0 & \rho_1 & 0 & 0 & \rho_2 \end{bmatrix}$$

$$\{\boldsymbol{P}\} = \begin{Bmatrix} N_{x,x}^T + N_{xy,y}^T - (1/2R)M_{xy,y}^T \\ N_{xy,x}^T + N_{y,y}^T + (1/2R)M_{xy,x}^T \\ -p - N_y^T/R \\ M_{x,x}^T + M_{xy,y}^T \\ M_{xy,x}^T + M_{y,y}^T \end{Bmatrix} \tag{10}$$

Note that in the present problem $N_{xy}^T = M_{xy}^T = 0$ and the vector $\{\boldsymbol{P}\}$ is simplified accordingly.

Boundary conditions (7) become

$$x = 0, \qquad x = L: \qquad w = \psi_y = 0$$

$$\left[B_{11}\frac{\partial}{\partial x} + \left(B_{16} - \frac{D_{16}}{2R}\right)\frac{\partial}{\partial y}\right]u + \left[\left(B_{16} + \frac{D_{16}}{2R}\right)\frac{\partial}{\partial x} + B_{12}\frac{\partial}{\partial y}\right]v$$

$$+ \frac{B_{12}}{R}w + \left(D_{11}\frac{\partial}{\partial x} + D_{16}\frac{\partial}{\partial y}\right)\psi_x$$

$$+ \left(D_{16}\frac{\partial}{\partial x} + D_{12}\frac{\partial}{\partial y}\right)\psi_y = M_x^T$$

$$\left(A_{11}\frac{\partial}{\partial x} - \frac{B_{16}}{2R}\frac{\partial}{\partial y}\right)u + \left(\frac{B_{16}}{2R}\frac{\partial}{\partial x} + A_{12}\frac{\partial}{\partial y}\right)v + \frac{A_{12}}{R}w$$

$$+ \left(B_{11}\frac{\partial}{\partial x} + B_{16}\frac{\partial}{\partial y}\right)\psi_x + \left(B_{16}\frac{\partial}{\partial x} + B_{12}\frac{\partial}{\partial y}\right)\psi_y = \bar{N}_x + N_x^T$$

$$\left(A_{66} - \frac{B_{66}}{2R}\right)\frac{\partial u}{\partial y} + \left(A_{66} + \frac{B_{66}}{2R}\right)\frac{\partial v}{\partial x} + \left(B_{16}\frac{\partial}{\partial x} + B_{66}\frac{\partial}{\partial y}\right)\psi_x$$

$$+ \left(B_{66}\frac{\partial}{\partial x} + B_{26}\frac{\partial}{\partial y}\right)\psi_y + \frac{3}{2R}\left\{\left[B_{16}\frac{\partial}{\partial x} + \left(B_{66} - \frac{D_{66}}{2R}\right)\frac{\partial}{\partial y}\right]u\right.$$

$$+ \left[\left(B_{66} + \frac{D_{66}}{2R}\right)\frac{\partial}{\partial x} + B_{26}\frac{\partial}{\partial y}\right]v + B_{26}\frac{w}{R}$$

$$\left. + \left(D_{16}\frac{\partial}{\partial x} + D_{66}\frac{\partial}{\partial y}\right)\psi_x + \left(D_{66}\frac{\partial}{\partial x} + D_{26}\frac{\partial}{\partial y}\right)\psi_y\right\} = \bar{N}_{xy} \tag{11}$$

The simplifications for plane sections of the shell ($R = \infty$) are straightforward.

Nonlinear Local Deformations of Thin Facings

As already indicated, the facings are assumed thin and in the state of plane stress. If a nonlinear problem is considered employing the coordinate system of the deformed middle surface, the equilibrium equations coincide with those of the linear shell theory.[13] Accordingly, Eqs. (1) are applicable in the nonlinear analysis. However, following the approach of Simitses et al.,[19] the underlined terms in the first two equations are omitted, i.e., the rotations about the normal to the middle surface are disregarded. The third equilibrium equation should be modified by incorporating the reaction of a nonlinear elastic foundation. The elastic foundation introduces the stiffness of the opposite facing into the analysis. If the core can be assumed infinitely stiff, as is often the case where the core is metallic, it only transfers normal forces between the facings. However, if the core is made of foam or honeycomb, its stiffness will affect the parameters of the equivalent elastic foundation. In this case the compliance of the opposite facing should be added to that of the core by assuming that they work in series. Note that extensive work on the characterization of effective core properties was referred to in the review of Noor et al.[8]

The evaluation of the stiffness of the elastic foundation can be carried out by considering the opposite facing with or without the core as an independent shell subjected to a local normal concentrated force. This analysis can be conducted using the approach presented in this section, without accounting for an elastic foundation. Note that the stiffness determined as a result of the analysis will be a function of the coordinates x and y. The fact that cylindrical shells and flat plates are characterized by a quadratic-cubic nonlinearity (see Appendix B) enables us to assume the reaction of the foundation in the form

$$F(x, y) = k_1 w + k_2 w^2 + k_3 w^3 \tag{12}$$

where the coefficients $k_i(x, y)$ should be obtained analytically (for example, the Rayleigh–Ritz method could be used to solve this problem), experimentally, or from the finite element analysis.

The three equations of motion, obtained from Eqs. (1), according to Sanders,[13] become

$$N_{x,x} + N_{xy,y} = \rho_f \ddot{u}$$

$$N_{xy,x} + N_{y,y} - (N_y/R)(v/R - w_{,y}) + (N_{xy}/R)w_{,x}$$

$$+ (1/R)(M_{xy,x} + M_{y,y}) = \rho_f \ddot{v}$$

$$(N_x w_{,x})_{,x} - [N_{xy}(v/R - w_{,y})]_{,x} + (N_{xy} w_{,x})_{,y} - [N_y(v/R - w_{,y})]_{,y}$$

$$- N_y/R + M_{x,xx} + 2M_{xy,xy} + M_{y,yy}$$

$$= k_1 w + k_2 w^2 + k_3 w^3 - p + \rho_f \ddot{w} \tag{13}$$

The solution of these equations should satisfy the boundary conditions along $x = 0$ and L, corresponding to a simply supported facing loaded by axial and shear loadings:

$$x = 0, \qquad x = L: \qquad w = M_x = 0$$

$$N_x = \bar{N}_x, \qquad N_{xy} + \tfrac{3}{2}(M_{xy}/R) = \bar{N}_{xy} \qquad (\bar{M}_{xy} = 0) \qquad (14)$$

The strain-displacement relationships for a thin shell can be derived from the general nonlinear theory.[25] Following the assumptions that the strains remain small, rotations about in-plane axes are moderate, whereas the rotations about the normal to the surface are negligible; these relationships are reduced to the form

$$\varepsilon_x = \varepsilon_x^o + z\kappa_x, \qquad \varepsilon_y = \varepsilon_y^o + z\kappa_y, \qquad \gamma_{xy} = \gamma_{xy}^o + z\kappa_{xy}$$

$$\varepsilon_x^o = u_{,x} + \tfrac{1}{2}w_{,x}^2, \qquad \varepsilon_y^o = v_{,y} + w/R + \tfrac{1}{2}(v/R - w_{,y})^2$$

$$\gamma_{xy}^o = u_{,y} + v_{,x} + w_{,x}(w_{,y} - v/R), \qquad \kappa_x = -w_{,xx}$$

$$\kappa_y = v_{,y}/R - w_{,yy}, \qquad \kappa_{xy} = -w_{,xy} + v_{,x}/2R \qquad (15)$$

where z is the distance from the middle surface of the facing. Note that Eqs. (15) were also presented by Sanders.[13]

The constitutive relations have to be modified to account for the symmetry of the facing. Introducing the stiffness matrices calculated using the middle surface of the multilayer facing as a reference surface, these relations become

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 & & & \\ A_{12} & A_{22} & 0 & & 0 & \\ 0 & 0 & A_{66} & & & \\ & & & D_{11} & D_{12} & 0 \\ & 0 & & D_{12} & D_{22} & 0 \\ & & & 0 & 0 & D_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \gamma_{xy}^o \\ \kappa_x \\ \kappa_y \\ 2\kappa_{xy} \end{Bmatrix} - \begin{Bmatrix} N_x^T \\ N_y^T \\ 0 \\ M_x^T \\ M_y^T \\ 0 \end{Bmatrix} \qquad (16)$$

A thermal gradient in the thickness direction results in a nonuniform degradation of material properties. In this case the symmetry about the middle surface of the facing is violated, and Eqs. (16) have to be replaced with a more complicated version accounting for coupling.

Now the equations of motion can be written in terms of displacements:

$$N_1(u, v, w) = \rho_f \ddot{u} + N_{x,x}^T$$

$$N_2(u, v, w) = \rho_f \ddot{v} + N_{y,y}^T + (1/R)M_{y,y}^T$$

$$N_3(u, v, w) = \rho_f \ddot{w} + M_{x,xx}^T + M_{y,yy}^T - (N_y^T/R) + k_1 w$$

$$+ k_2 w^2 + k_3 w^3 - p \qquad (17)$$

where $N_i(u, v, w)$ are nonlinear differential operators given in Appendix B, and the thermal contributions are calculated using equations similar to Eqs. (6) where the integration is performed over the thickness of the facing under consideration and z is the distance from the middle surface of this facing.

Thermomechanical Coupling at the Micromechanical Level

The solution methodology that should be implemented for the analysis of a structure subjected to thermomechanical loading is reflected in Fig. 2. Link 1 reflects the effect of temperature and stress on the material properties, link 2 accounts for the effect of stress on temperature within the material, and links 3 and 4 are related to the macromechanical problems of heat transfer (3) and to the stress problem (4). Link 5 reflects both macromechanical as well as micromechanical phenomena related to the effect of material properties on a temperature distribution. An appropriate micromechanics block is not explicitly shown in Fig. 2, although it is implicitly included in the bullet Properties. As follows from Fig. 2, the problem of thermomechanical stresses is inheritantly nonlinear, and coupling is present at all phases of the solution. This means that the solution should be iterative, including adjustments of the properties and temperature at each step and consequent reevaluation of the stresses.

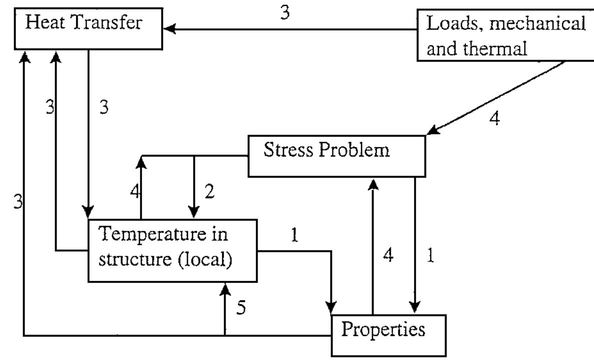

Fig. 2 Mutual interaction of elements of a thermomechanical problem. If the problem is adiabatic, heat transfer and the corresponding links are absent.

In this section the effects of stress on temperature and the influence of temperature and stress on the composite material properties are briefly discussed, i.e., links 1 and 2 and 5 (micromechanics) in Fig. 2. In particular, an increase in temperature of an isotropic material subjected to a multidimensional stress field was derived in the form[16]

$$\Delta T = -\frac{T}{\rho C_\sigma}\left\{\left[\alpha_0 + \left(\frac{\nu}{E^2}\frac{\partial E}{\partial T} - \frac{1}{E}\frac{\partial \nu}{\partial T}\right)s\right]\Delta s - \sum_{i=1}^{3}\left(\frac{1+\nu}{E^2}\frac{\partial E}{\partial T} - \frac{1}{E}\frac{\partial \nu}{\partial T}\right)\sigma_{ii}\,\Delta\sigma_{ii}\right\} \qquad (18)$$

Note that Eq. (18) corresponds to adiabatic conditions, where heat transfer may be disregarded.

Equation (18) can be applied to the materials of the fibers and the matrix. However, the modulus of elasticity and the Poisson ratio of the fibers are often insensitive to moderate variations of temperature. In this case Eq. (18) for the fibers is simplified:

$$\Delta T_f = -(T/\rho_f C_\sigma^f)\alpha_0^f \Delta s \qquad (19)$$

Environmental effects on the properties of polymeric matrices have been investigated by Chamis and his collaborators.[14] In particular, the modulus of elasticity of the matrix can be calculated by

$$E_m = E_{m0}\left(\frac{T_{gw} - T}{T_{g0} - T_0}\right)^{\frac{1}{2}} \qquad (20)$$

where T_0 is a reference temperature and E_{m0} is the modulus of elasticity in the absence of degradation. The Poisson ratio of a polymeric matrix is usually not affected appreciably by temperature and can be assumed constant. A derivative of the matrix modulus of elasticity with respect to temperature follows from Eq. (20):

$$\frac{\partial E_m}{\partial T} = -\frac{E_{m0}}{2[(T_{gw} - T)(T_{g0} - T_0)]^{\frac{1}{2}}} \qquad (21)$$

Equation (21) can be substituted into Eq. (18), yielding the change of temperature in the matrix, i.e., ΔT_m. Now the change of the layer temperature can be calculated by[18]

$$\Delta T_{p\ell y} = \frac{V_f \rho_f C_\sigma^f \Delta T_f + V_m \rho_m C_\sigma^m \Delta T_m}{V_f \rho_f C_\sigma^f + V_m \rho_m C_\sigma^m} \qquad (22)$$

Whereas the solution given by Eq. (22) accounts for the changes of temperature caused by a thermomechanical coupling, the other result of this coupling is the change of the material properties. In particular, the coefficient of thermal expansion of an isotropic material is affected by the stresses and temperature as is shown in the term in the brackets on the right side of Eq. (18). Accordingly, the adiabatic analysis of a shell subjected to a dynamic loading can be conducted as follows:

1) At each time instant the stresses are evaluated using the properties of the material obtained at the preceding time interval. The new values of stresses are used to update the values of the coefficients of thermal expansion of the fibers and the matrix throughout the material. Simultaneously, the modulus of elasticity of the matrix is adjusted according to Eq. (20). The shear modulus should also be adjusted using a similar equation. The updated properties of the composite material can be used in an appropriate micromechanical theory to update the engineering constants of the composite material. Note that these constants may be nonuniform throughout the material, as a result of a nonuniform temperature distribution. Even if temperature is uniform, the coefficients of thermal expansion will be affected by local stresses, so that they will vary throughout the material.

2) The temperature increase caused by thermomechanical coupling can be calculated using Eqs. (18), (19), and (22), although this increase may prove negligible in practical applications.

3) The stresses at the end of the time interval can be reevaluated using the macromechanical solution and the adjusted material constants. Note that the variations of the material properties with time imply that these properties depend on the coordinates. This would result in a very complicated formulation. Therefore, continuously varying properties dependent on the coordinates can be replaced with a section-wise instantaneously constant-averaged properties so that the macromechanical equations introduced in the preceding sections remain valid. Then the process is repeated for the next time interval, etc.

Conclusions

The paper presents an outline of a comprehensive analysis of sandwich box-type shell structures subjected to a combination of mechanical and thermal loads. The first part of the paper deals with the global deformation and stress problem for a sandwich with dissimilar facings using a shear-deformable version of Sanders's shell theory. Subsequently, governing equations for nonlinear deformations of a facing are considered using the nonlinear Sanders shell theory. The support of the opposite facing and the core is represented by a nonlinear elastic foundation. The final part of the paper suggests an approach that can be used to incorporate thermomechanical coupling and its effect on the material properties and temperature in adiabatic problems.

Appendix A: Elements of the Matrix of Linear Differential Operators [L] in Equations (8)

$$L_{11} = A_{11}\frac{\partial^2}{\partial x^2} - \frac{B_{16}}{R}\frac{\partial^2}{\partial x\partial y} + \left(A_{66} - \frac{B_{66}}{R} + \frac{D_{66}}{4R^2}\right)\frac{\partial^2}{\partial y^2}$$

$$L_{12} = \frac{B_{16}}{2R}\frac{\partial^2}{\partial x^2} + \left(A_{12} + A_{66} - \frac{D_{66}}{4R^2}\right)\frac{\partial^2}{\partial x\partial y} - \frac{B_{26}}{2R}\frac{\partial^2}{\partial y^2}$$

$$L_{13} = \frac{A_{12}}{R}\frac{\partial}{\partial x} - \frac{B_{26}}{2R^2}\frac{\partial}{\partial y}$$

$$L_{14} = B_{11}\frac{\partial^2}{\partial x^2} + \left(2B_{16} - \frac{D_{16}}{2R}\right)\frac{\partial^2}{\partial x\partial y} + \left(B_{66} - \frac{D_{66}}{2R}\right)\frac{\partial^2}{\partial y^2}$$

$$L_{15} = B_{16}\frac{\partial^2}{\partial x^2} + \left(B_{12} + B_{66} - \frac{D_{66}}{2R}\right)\frac{\partial^2}{\partial x\partial y} + \left(B_{26} - \frac{D_{26}}{2R}\right)\frac{\partial^2}{\partial y^2}$$

$$L_{21} = L_{12}$$

$$L_{22} = \left(A_{66} + \frac{B_{66}}{R} + \frac{D_{66}}{4R^2}\right)\frac{\partial^2}{\partial x^2} + \frac{B_{26}}{R}\frac{\partial^2}{\partial x\partial y} + A_{22}\frac{\partial^2}{\partial y^2} - \frac{k_{44}A_{44}}{R^2}$$

$$L_{23} = \frac{B_{26}}{2R^2}\frac{\partial}{\partial x} + \frac{A_{22} + k_{44}A_{44}}{R}\frac{\partial}{\partial y}$$

$$L_{24} = \left(B_{16} + \frac{D_{16}}{2R}\right)\frac{\partial^2}{\partial x^2} + \left(B_{12} + B_{66} + \frac{D_{66}}{2R}\right)\frac{\partial^2}{\partial x\partial y} + B_{26}\frac{\partial^2}{\partial y^2}$$

$$L_{25} = \frac{k_{44}A_{44}}{R} + \left(B_{66} + \frac{D_{66}}{2R}\right)\frac{\partial^2}{\partial x^2} + \left(2B_{26} + \frac{D_{26}}{2R}\right)\frac{\partial^2}{\partial x\partial y} + B_{22}\frac{\partial^2}{\partial y^2}$$

$$L_{31} = -L_{13}, \qquad L_{32} = -L_{23}$$

$$L_{33} = -\frac{A_{22}}{R^2} + k_{55}A_{55}\frac{\partial^2}{\partial x^2} + k_{44}A_{44}\frac{\partial^2}{\partial y^2}$$

$$L_{34} = \left(k_{55}A_{55} - \frac{B_{12}}{R}\right)\frac{\partial}{\partial x} - \frac{B_{26}}{R}\frac{\partial}{\partial y}$$

$$L_{35} = -\frac{B_{26}}{R}\frac{\partial}{\partial x} + \left(k_{44}A_{44} - \frac{B_{22}}{R}\right)\frac{\partial}{\partial y}$$

$$L_{41} = L_{14}, \qquad L_{42} = L_{24}, \qquad L_{43} = -L_{34}$$

$$L_{44} = -k_{55}A_{55} + D_{11}\frac{\partial^2}{\partial x^2} + 2D_{16}\frac{\partial^2}{\partial x\partial y} + D_{66}\frac{\partial^2}{\partial y^2}$$

$$L_{45} = D_{16}\frac{\partial^2}{\partial x^2} + (D_{12} + D_{16})\frac{\partial^2}{\partial x\partial y} + D_{26}\frac{\partial^2}{\partial y^2}$$

$$L_{51} = L_{15}, \qquad L_{52} = L_{25}, \qquad L_{53} = -L_{35}, \qquad L_{54} = L_{45}$$

$$L_{55} = -k_{44}A_{44} + D_{66}\frac{\partial^2}{\partial x^2} + 2D_{26}\frac{\partial^2}{\partial x\partial y} + D_{22}\frac{\partial^2}{\partial y^2}$$

Appendix B: Nonlinear Functions in Equations (17)

$$N_i(u, v, w) = N_i'(u, v, w) + N_i''(u, v, w) + N_i'''(u, v, w)$$

where N_i', N_i'', and N_i''' are linear, quadratic, and cubic functions, respectively:

$$N_1'(u, v, w) = A_{11}u_{,xx} + A_{66}u_{,yy} + (A_{12} + A_{66})v_{,xy} + \frac{A_{12}}{R}w_{,x}$$

$$N_2'(u, v, w) = (A_{12} + A_{66})u_{,xy} + \frac{N_y^T}{R^2}v + \left(A_{66} + \frac{D_{66}}{R^2}\right)v_{,xx} + \left(A_{22} + \frac{D_{22}}{R^2}\right)v_{,yy} + \frac{A_{22} - N_y^T}{R}w_{,y} - \frac{D_{12} + 2D_{66}}{R}w_{,xxy} - \frac{D_{22}}{R}w_{,yyy}$$

$$N_3'(u, v, w) = -\frac{A_{12}}{R}u_{,x} - \frac{A_{22}}{R}\left(v_{,y} + \frac{w}{R}\right) - D_{11}w_{,xxxx} - (D_{12} + 2D_{66})\left(2w_{,xxyy} - \frac{v_{,xxy}}{R}\right) - D_{22}\left(w_{,yyyy} - \frac{v_{,yyy}}{R}\right) - (N_x^T w_{,x})_{,x} - \left[N_y^T\left(w_{,y} - \frac{v}{R}\right)\right]_{,y}$$

$$N_1''(u, v, w) = (A_{11}w_{,xx} + A_{66}w_{,yy})w_{,x} + (A_{12} + A_{66})w_{,xy}w_{,y} - \frac{1}{R}[(A_{12} + A_{66})vw_{,xy} + A_{12}v_{,x}w_{,y} + A_{66}v_{,y}w_{,x}] + \frac{A_{12}}{R^2}vv_{,x}$$

$$N_2''(u, v, w) = \frac{1}{R}\left(A_{12}u_{,x}w_{,y} - \frac{A_{12}}{R}u_{,x}v + A_{66}u_{,y}w_{,x}\right) + (A_{12} + A_{66})w_{,x}w_{,xy} + A_{22}w_{,y}w_{,yy} + A_{66}w_{,y}w_{,xx} + \frac{A_{22}}{R^2}ww_{,y} - \frac{A_{66}}{R}vw_{,xx} - \frac{A_{22}}{R}\left(w_{,yy} + \frac{w}{R^2}\right)v$$

$$N_3''(u,v,w) = \left\{\left[A_{11}u_{,x} + A_{12}\left(v_{,y} + \frac{w}{R}\right)\right]w_{,x}\right\}_{,x} + \left[A_{66}(u_{,y} + v_{,x})\left(w_{,y} - \frac{v}{R}\right)\right]_{,x} + [A_{66}(u_{,y} + v_{,x})w_{,x}]_{,y} + \left\{\left[A_{12}u_{,x} + A_{22}\left(v_{,y} + \frac{w}{R}\right)\right]\left(w_{,y} - \frac{v}{R}\right)\right\}_{,y} - \frac{1}{R}\left\{\frac{A_{12}}{2}w_{,x}^2 + A_{22}\left[\frac{1}{2}\left(\frac{v}{R}\right)^2 - \frac{v}{R}w_{,y} + \frac{1}{2}w_{,y}^2\right]\right\}$$

$$N_1'''(u,v,w) = 0$$

$$N_2'''(u,v,w) = \left\{\left(\frac{A_{12}}{2} + A_{66}\right)w_{,x}^2 + A_{22}\left[\frac{1}{2}\left(\frac{v}{R}\right)^2 - \frac{v}{R}w_{,y} + \frac{1}{2}w_{,y}^2\right]\right\}\left(w_{,y} - \frac{v}{R}\right)\frac{1}{R}$$

$$N_3'''(u,v,w) = \left(\left\{\frac{A_{11}}{2}w_{,x}^2 + A_{12}\left[\frac{1}{2}\left(\frac{v}{R}\right)^2 - \frac{v}{R}w_{,y} + \frac{1}{2}w_{,y}^2\right]\right\}w_{,x}\right)_{,x} + A_{66}\left\{\left[\left(w_{,y} - \frac{v}{R}\right)^2 w_{,x}\right]_{,x} + \left[\left(w_{,y} - \frac{v}{R}\right)w_{,x}^2\right]_{,y}\right\} + \left(\left\{\frac{A_{12}}{2}w_{,x}^2 + A_{22}\left[\frac{1}{2}\left(\frac{v}{R}\right)^2 - \frac{v}{R}w_{,y} + \frac{1}{2}w_{,y}^2\right]\right\}\left(w_{,y} - \frac{v}{R}\right)\right)_{,y}$$

Acknowledgments

This work has been supported by the U.S. Air Force Office of Scientific Research, Contract F49620098-2-0384. The program managers are Brian Sanders and Ozden Ochoa. The authors appreciate useful comments of the reviewers.

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